Chapter 13 GMM for Linear Factor Models in Discount Factor form

- GMM on the pricing errors gives a cross-sectional regression
- The case of excess returns
- Horse race
- Testing for characteristic
- Testing for priced factors: lambdas or b's

13.1 GMM on the pricing errors Gives a Cross-sectional regression

- The asset pricing model is :

$$m = b'f, \qquad E(p) = E(mx)$$

or simply $\quad E(p) = E(xf')b \qquad (13.1)$

in vector form

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_K \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix}$$

- Apply the pricing errors as GMM moments

$$g_T(b) = E_T(xf'b - p)$$

- The GMM estimate is formed from:

$$\min_b g_T(b)' W g_T(b)$$

with first-order condition

$$d'Wg_T(b) = d'WE_T(xf'b - p) = 0$$

where

$$d' = \frac{\partial g_T(b)'}{\partial b} = E_T(fx')$$

- The first stage has $W = I$,the second stage has $W = S^{-1}$.The GMM estimate is

$$fisrstage : \hat{b}_1 = (d'd)^{-1} d'E_T(p)$$

$$\sec ondstage : \hat{b}_2 = \left(d'S^{-1}d\right)^{-1} d'S^{-1}E_T(p)$$

- The first-stage estimate is an OLS cross-sectional regression of average prices on the second moment of payoff with factors, and the second-stage is a GLS cross-sectional regression
- Easily shown in (13.1)

- The standard error (from 11.2,11.8):

$$fisrtstage : \operatorname{cov}\left(\hat{b}_1\right) = \frac{1}{T}(d'd)^{-1} d'Sd\,(d'd)^{-1}$$

$$\sec ondstage : \operatorname{cov}\left(\hat{b}_2\right) = \frac{1}{T}\left(d'S^{-1}d\right)^{-1}$$

- The covariance matrix of the pricing errors is (from 11.5,11.9,11.10)

$$fisrtstage : T\operatorname{cov}[g_T\left(\hat{b}_1\right)] = (I - d\,(d'd)^{-1} d')S(I - d\,(d'd)^{-1} d')$$

$$\sec ondstage : T\operatorname{cov}[g_T\left(\hat{b}_2\right)] = S - d\left(d'S^{-1}d\right)^{-1} d'$$

- The model test:

$$g_T\left(\hat{b}\right)'\operatorname{cov}(g_T\left(\hat{b}\right))^{-1}g_T\left(\hat{b}\right)\sim\chi^2(\#\,moments-\#\,parameters)$$

which specializes for the second-stage estimate is :

$$Tg_T\left(\hat{b}\right)'S^{-1}g_T\left(\hat{b}\right)\sim\chi^2(\#\,moments-\#\,parameters)$$

it turns out that the χ^2 test has the same value for first and second stage

13.2 The Case of Excess Returns

- If all assets are excess returns , the model is lack of identification.
- Write the model as

$$m = a - b' f$$

normalize a=1, then

$$g_T(b) = E_T(mR^e) = E_T(R^e) - E_T(R^e f')b$$

we have:

$$d' = \frac{\partial g_T(b)}{\partial b} = E_T(fR^{e\prime})$$

the first-order condition is

$$d'W\left(db - E_T\left(R^e\right)\right) = 0$$

the GMM estimates of b are

$$fisrstage : \hat{b}_1 = \left(d'd\right)^{-1} d'E_T\left(R^e\right)$$

$$\sec ondstage : \hat{b}_2 = \left(d'S^{-1}d\right)^{-1} d'S^{-1}E_T\left(R^e\right)$$

- The GMM estimate is a cross-sectional regression of mean excess returns on the second moments of returns with factors

Mean returns on covariance

- We can obtain a cross-sectional regression of mean returns on covariance
- Normalize $a = 1 + b'E(f)$ rather than a=1

then, the model is :

$$m = 1 - b'\left(f - E(f)\right) = 1 - b'\tilde{f}$$

with $E(m) = 1$, the pricing errors are:

$$g_T(b) = E_T\left(mR^e\right) = E_T\left(R^e\right) - E_T\left(R^e \tilde{f}'\right)b$$

$$d' = \frac{\partial g_T(b)}{\partial b'} = E_T\left(\tilde{f}R^{e\prime}\right) = \mathrm{cov}_T\left(R^e, f\right)$$

the first-order condition is:

$$d'W\left[db - E_T\left(R^e\right)\right] = 0$$

The GMM estimates of b are:

$$fisrstage: \hat{b}_1 = (d'd)^{-1} d'E_T\left(R^e\right)$$

$$\sec ondstage: \hat{b}_2 = \left(d'S^{-1}d\right)^{-1} d'S^{-1}E_T\left(R^e\right)$$

- The GMM estimate is a cross-sectional regression of expected returns on the covariance between returns and factors

- The standard errors and variance of the pricing errors are the same as in (13.2) (13.3),with different d matrix.
- The $p = E(mx)$ for excess return is equivalent to $E(R^e) = -\mathrm{cov}(R^e, f')b$ thus the covariance enter in place of betas

$$
\begin{aligned}
0 &= E(m)E(R^e) + \mathrm{cov}(m, R^e) \\
&= E(R^e) + \mathrm{cov}(m, R^e) \\
&= E(R^e) + \mathrm{cov}(R^e, f')b
\end{aligned}
$$

- There is one fly in the ointment, the mean of the factor $E(f)$ is estimated. The distribution should recognize sampling variance.
- It is better to use some other non-sample-dependent normalization for a.

13.3 Horse Races

- How to test whether one set of factors drives another.
- The general model is :

$$m = b_1' f_1 + b_2' f_2$$

- Two methods to test $b_2 = 0$?

1. Wald test.

$$\hat{b}_2' \operatorname{var}(\hat{b}_2)^{-1} \hat{b}_2 \sim \chi^2_{\# b_2}$$

2. χ^2 Difference test

restrict condition: $b_2 = 0$

$$TJ_T(restricted) - TJ_T(unrestricted) \sim \chi^2(\#ofrestrictions)$$

This is very much like a likelihood ratio test

13.4 Testing for Characteristics

- In a good asset pricing model, the alphas or pricing errors should drive out the characteristic.
- Denote the characteristic of portfolio y_i by i. Let y_t^i denote the time series whose mean $E\left(y_t^i\right)$ determines the characteristic

write the moment condition for the assets as:

$$g_T = E_T \left(m_{t+1}(b) x_{t+1} - p_t - \gamma y_t \right)$$

The GMM estimate of γ is:

$$\hat{\gamma} = \left(E_T(y)' W E_T(y) \right) E_T(y)' W g_T$$

To test whether $\gamma = 0$ is statistically significant.

13.5 Testing for priced Factors:Lambdas or b’s ?

- b_j asks whether factor j helps to price assets given other factors.

$$m = b' f = b_1 f_1 + b_2 f_2 + \cdots + b_j f_j + \cdots + b_K f_K$$

- b_j is the *multiple* regression coefficient of m on f_j given all the other factors.
- λ_j asks whether factor j is priced, or whether its factor-mimicking portfolio carries a positive risk premium

to explain these , use mean-zero factors, excess returns , normalize $E(m)=1$

see Section 6.3 : $m = 1 - f'b$

$$0 = E(mR^e) = E\left[R^e(1 - f'b)\right]$$

$$E(R^e) = E(R^e f'b) = \text{cov}(R^e, f')b = \text{cov}(R^e, f')E(ff')^{-1}E(ff')b = \beta\lambda$$

$$\lambda = E(ff')b$$

write

$$\lambda = E\left[f(f'b)\right] = E\left[f(1-m)\right] = -E(mf)$$

So λ_j captures whether factor f_j is priced

- λ_j is proportional to the <u>*single*</u> regression coefficient of m on f

$$\lambda_j = -E\left(mf_j\right) = -\operatorname{cov}\left(m, f_j\right) = -\frac{\operatorname{cov}\left(m, f_j\right)}{\operatorname{var}\left(f_j\right)}\operatorname{var}\left(f_j\right)$$

- $\lambda_j = 0$ ask “is the factor f_j correlated with the true discount factor?”
- When the factors are orthogonal,

$$\lambda_j = 0 \Leftrightarrow b_j = 0 \qquad \lambda = E\left(ff'\right)b$$

- If you want to know whether factor i is priced , look at $\lambda\left(or E\left(m f_i\right)\right)$
- If you want to know whether factor i helps to price assets, look at b_i
- For example , suppose CAPM is true,

$$m = a - bR^{em}$$

consider another R^{ex} (positively correlated with R^{em})as spurious factor

The answer is :

$$m = a - bR^{em} - b_x R^{ex}$$

$b_x = 0$ indicating that this factor does not help to price assets.

however

$$E\left(R^{ei}\right) = \beta_{im}\lambda_m + \beta_{ix}\lambda_x$$

Mean-Variance Frontier and Performance Evalution

- Stock returns from many countries are not perfectly correlated, so it looks like one can reduce portfolio variance by holding an internationally diversified portfolio.
- Is it real or just sampling errors?
- When evaluating fund manager, we want to know whether the manager is truly able to form a portfolio that beats mean-variance efficient portfolios, or just due to luck.
- A factor model is true if only if a linear combination of the factors is mean-variance efficient
- A CAPM, $p = E(mx), m = a - bR^p$ test analogously tests whether R^p is on the mean-variance frontier of the test assets.

- To test whether the R^d assets span the mean-variance frontier of R^d and R^i

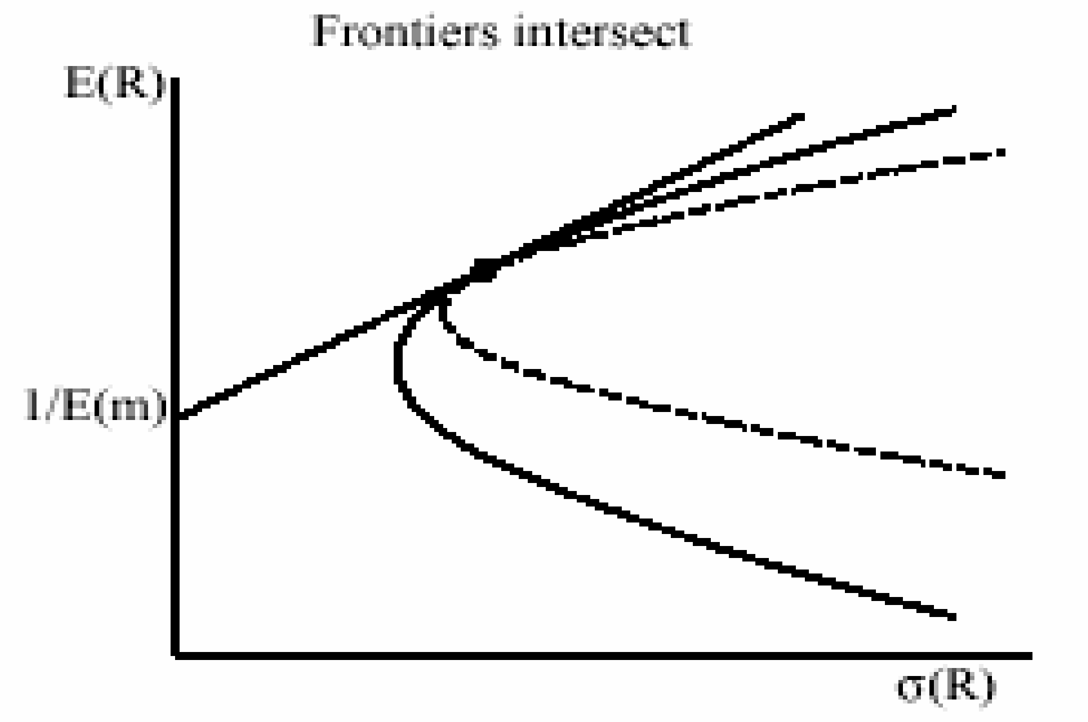

Figure 27. Mean variance frontiers might intersect rather than coincide.

- DeSantis (1992) and Chen and Knez (1992,1993):

for intersection, $m = a - b_d{}' R^d$ will price both R^d and R^f only for one value of a .

thus, we can test for coincident frontiers by testing whether $m = a - b_d{}' R^d$ prices both R^d and R^f for two prespecified values of a simulaneously

- If R^{d1} and R^{d2} on the frontier, there must be discount factors:

$$m^1 = a^1 - b^1 R^{d1}$$

$$m^2 = a^2 - b^2 R^{d2}$$

Test for spanning with a JT test, fixed a^1, a^2

$$E[(a^1 - b^1 R^{d1}) R^d] = 0$$

$$E[(a^1 - b^1 R^{d1}) R^i] = 0$$

$$E[(a^2 - b^2 R^{d2}) R^d] = 0$$

$$E[(a^2 - b^2 R^{d2}) R^i] = 0$$