Chapter 27
More on Numerical Procedures

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Numerical Procedures

Topics:

- Path dependent options using tree
- Barrier options
- Options where there are two stochastic variables
- American options using Monte Carlo

Path Dependence: The Traditional View

- Trees work well for American options. They cannot be used for path-dependent options
- Monte Carlo simulation works well for path-dependent options; it cannot be used for American options

Extending the Use of Trees

- Backwards induction can be used for some path-dependent options
- We will first illustrate the methodology using lookback options and then show how it can be used for Asian options

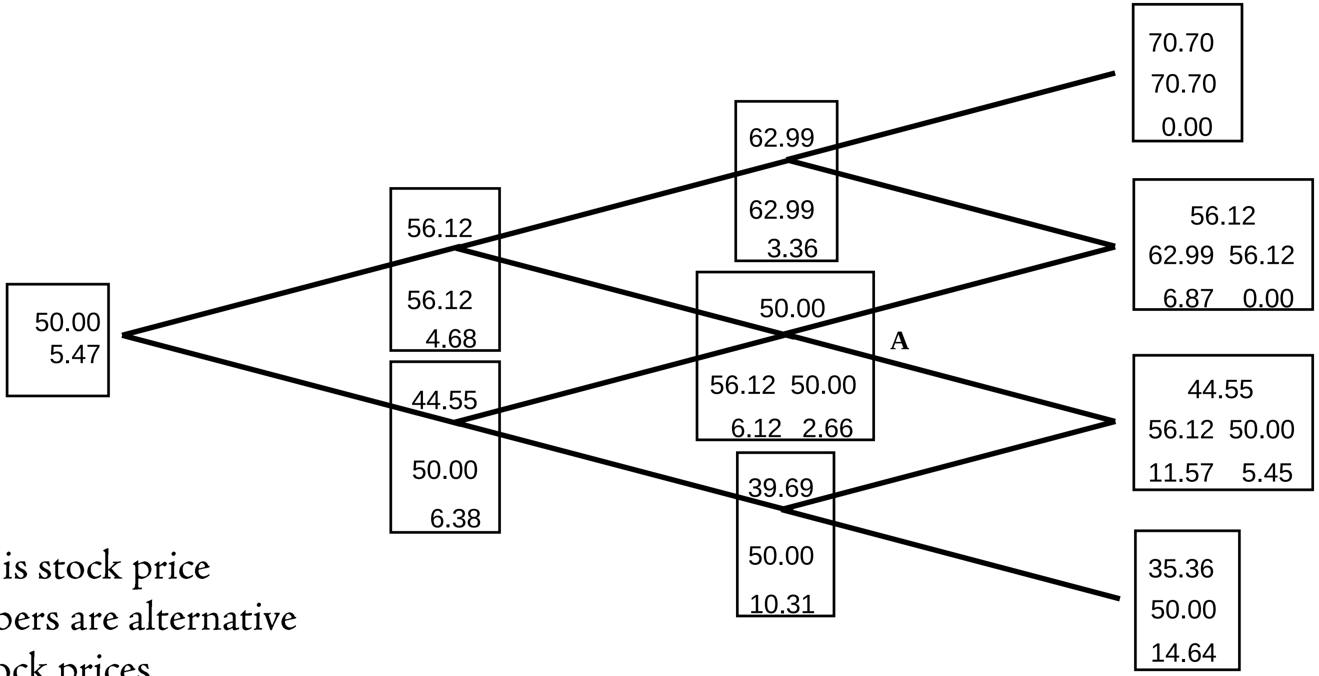
Lookback Example

- Consider an American lookback put on a stock where
  S = 50, s = 40%, r = 10%, Dt = 1 month & the life of the option is 3 months
- Payoff is S_{max}-S_T
- We can value the deal by considering all possible values of the maximum stock price at each node

(This example is presented to illustrate the methodology. It is not the most efficient way of handling American lookbacks (See Technical Note 13)

Example: An American Lookback Put Option

$S_0 = 50$, s = 40%, $r = 10\%$, Dt = 1 month,



Top number is stock price
Middle numbers are alternative maximum stock prices
Lower numbers are option prices

Why the Approach Works

This approach works for lookback options because

- The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a "path function")
- The value of the path function at a node can be calculated from the stock price at the node and from the value of the function at the immediately preceding node
- The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree

Extensions of the Approach

- The approach can be extended so that there are no limits on the number of alternative values of the path function at a node
- The basic idea is that it is not necessary to consider every possible value of the path function
- It is sufficient to consider a relatively small number of representative values of the function at each node

Working Forward

- First work forward through the tree calculating the max and min values of the "path function" at each node
- Next choose representative values of the path function that span the range between the min and the max
  - Simplest approach: choose the min, the max, and *N* equally spaced values between the min and max

Backwards Induction

- We work backwards through the tree in the usual way carrying out calculations for each of the alternative values of the path function that are considered at a node
- When we require the value of the derivative at a node for a value of the path function that is not explicitly considered at that node, we use linear or quadratic interpolation

Part of Tree to Calculate Value of an Option on the Arithmetic Average

Node Y: S = 54.68

Average S	Option Price
47.99	7.575
51.12	8.101
54.26	8.635
57.39	9.178

0.5056

Node X: S = 50.00

Average S	Option Price
46.65	5.642
49.04	5.923
51.44	6.206
53.83	6.492

0.4944

Node Z: S = 45.72

Average S	Option Price
43.88	3.430
46.75	3.750
49.61	4.079
52.48	4.416

S=50, X=50, σ=40%, r =10%, T=1yr, Δt=0.05yr. We are at time $4\Delta t$

Part of Tree to Calculate Value of an Option on the Arithmetic Average (continued)

Consider Node X when the average of 5 observations is 51.44

Node Y: If this is reached, the average becomes 51.98. The option price is interpolated as 8.247

Node Z: If this is reached, the average becomes 50.49. The option price is interpolated as 4.182

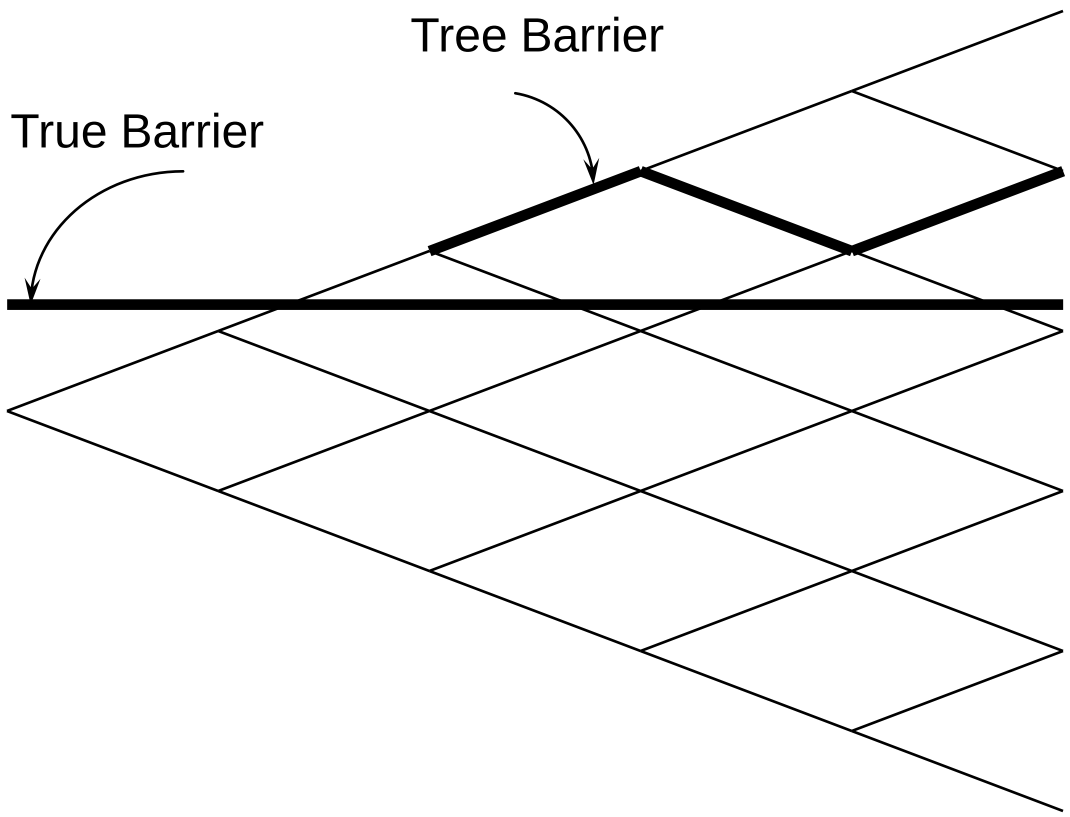
Node X: value is

$$(0.5056\times 8.247 + 0.4944\times 4.182)e^{-0.1\times 0.05} = 6.206$$

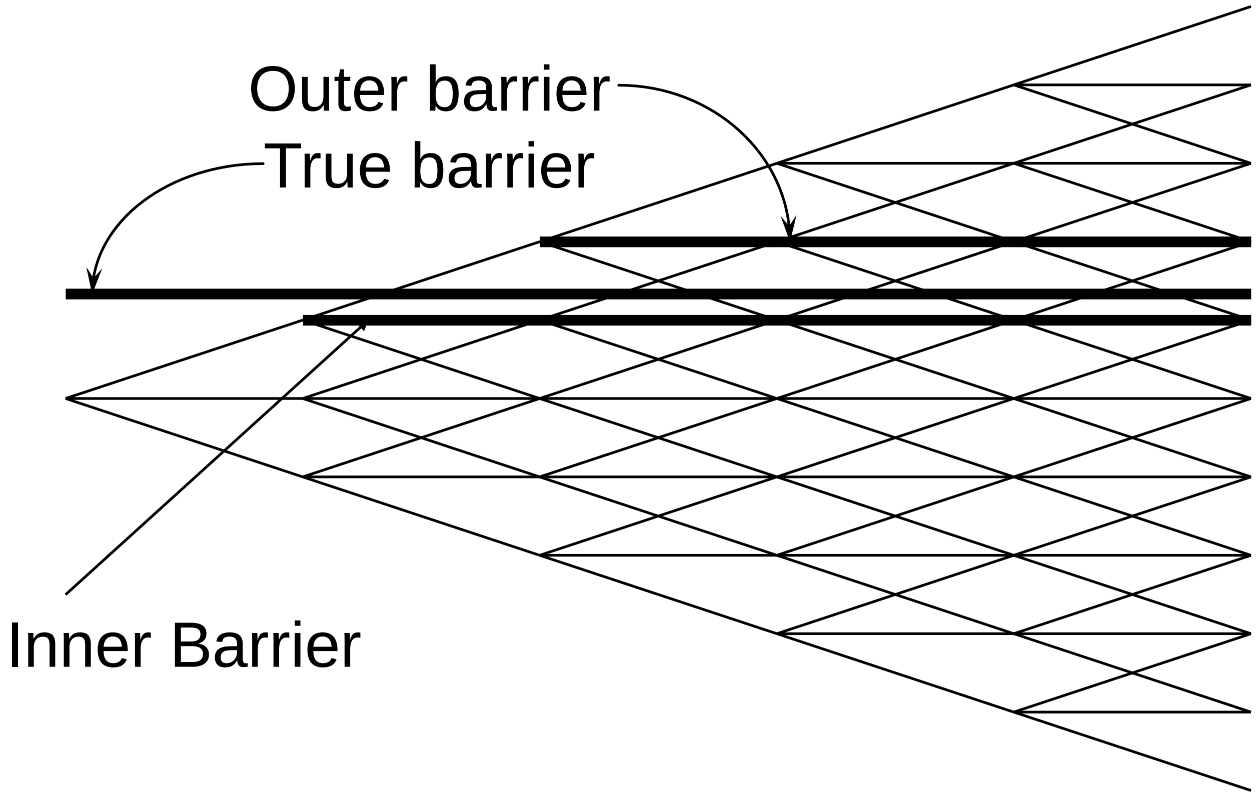
Using Trees with Barriers

- When trees are used to value options with barriers, convergence tends to be slow
- The slow convergence arises from the fact that the barrier is inaccurately specified by the tree

True Barrier vs Tree Barrier for a Knockout Option: The Binomial Tree Case



Inner and Outer Barriers for Trinomial Trees



Alternative Solutions to Valuing Barrier Options

- Interpolate between value when inner barrier is assumed and value when outer barrier is assumed
- Ensure that nodes always lie on the barriers
- Use adaptive mesh methodology

In all cases a trinomial tree is preferable to a binomial tree

Modeling Two Correlated Variables Using a 3-Dimensional Tree

- Approaches
  - Transform variables so that they are not correlated and build the tree in the transformed variables
  - Take the correlation into account by adjusting the position of the nodes
  - Take the correlation into account by adjusting the probabilities

Transforming Vaviables

- Suppose:

$$d \ln S_1 = \left(r - q_1 - \sigma_1^2 / 2\right) dt + \sigma_1 dz_1$$
$$d \ln S_2 = \left(r - q_2 - \sigma_2^2 / 2\right) dt + \sigma_2 dz_2$$
$$E\left(dz_1 dz_2\right) = \rho dt$$

- We define two new uncorrelated variables:

$$x_1 = \sigma_2 \ln S_1 + \sigma_1 \ln S_2$$
$$x_2 = \sigma_2 \ln S_1 - \sigma_1 \ln S_2$$

定理

- 定理：若dz_1和dz_2相关，则dz_1+dz_2与dz_1-dz_2不相关。
- 证明：由于E（dz_1+dz_2）和E（dz_1-dz_2）＝0，所以 $cov(dz_1+dz_2, dz_1-dz_2)=E(dz_1+dz_2)(dz_1-dz_2)=E(dz_1^2-dz_2^2)=dt-dt=0$

$$dx_1 = \left[\sigma_2\left(r - q_1 - \sigma_1^2/2\right) + \sigma_1\left(r - q_2 - \sigma_2^2/2\right)\right]dt$$

$$+\sigma_1\sigma_2\sqrt{2（1+\rho)}dz_A$$

$$dx_2 = \left[\sigma_2\left(r - q_1 - \sigma_1^2/2\right) - \sigma_1\left(r - q_2 - \sigma_2^2/2\right)\right]dt$$

$$+\sigma_1\sigma_2\sqrt{2（1-\rho)}dz_B$$

证明：

$$dz_1+dz_2=\sqrt{2\ \ 1+\rho\ }\,dz_A$$

$\because dz_1$和dz_2均为正态分布，其和也是正态分布。

它们之和的方差$=E\ \ dz_1+dz_2\ \ ^2=E\ \ dz_1^2+dz_2^2+2dz_1dz_2$

$=2(1+\rho)\ dt$

$$\therefore dz_1+dz_2=\sqrt{2\ \ 1+\rho\ \ dt}\times\varepsilon=\sqrt{2\ \ 1+\rho\ }\,dz_A$$

同理可证$dz_1-dz_2=\sqrt{2\ \ 1-\rho\ }\,dz_B$

Monte Carlo Simulation and American Options

- Two approaches:
  - The least squares approach
  - The exercise boundary parameterization approach
- Consider a 3-year put option where the initial asset price is 1.00, the strike price is 1.10, the risk-free rate is 6%, and there is no income

Sampled Paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The Least Squares Approach

- We work back from the end using a least squares approach to calculate the continuation value at each time
- Consider year 2. The option is in the money for five paths. These give observations on S of 1.08, 1.07, 0.97, 0.77, and 0.84. The continuation values are 0.00, $0.07e^{-0.06}$, $0.18e^{-0.06}$, $0.20e^{-0.06}$, and $0.09e^{-0.06}$

The Least Squares Approach continued

- Fitting a model of the form $V=a+bS+cS^2$ we get a best fit relation

$$V=-1.070+2.983S-1.813S^2$$

  for the continuation value V
- This defines the early exercise decision at $t=2$. We carry out a similar analysis at $t=1$

The Least Squares Approach continued

In practice more complex functional forms can be used for the continuation value and many more paths are sampled

The Early Exercise Boundary Parametrization Approach

- We assume that the early exercise boundary can be parameterized in some way
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.

Application to Example

- We parameterize the early exercise boundary by specifying a critical asset price, S^*, below which the option is exercised.
- At t =3 the optimal S^* for the eight paths is 1.10. At t =2 the optimal S^* is 0.84. At t =1 the optimal S^* is 0.88.
- In practice we would use many more paths to calculate the S^*