Chapter 21
Basic Numerical Procedures

郑振龙

厦门大学财务系

课程网站： http://efinance.org.cn

Email: zlzheng@xmu.edu.cn

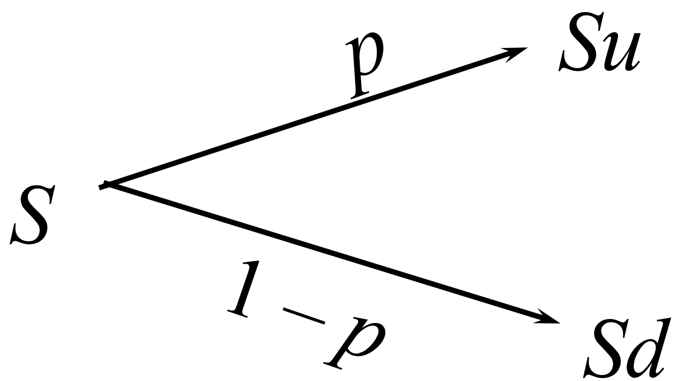
Approaches to Derivatives Valuation

- Trees
- Monte Carlo simulation
- Finite difference methods

Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset
- In each small interval of time the stock price is assumed to move up by a proportional amount u or to move down by a proportional amount d

Movements in Time Δt



Tree Parameters for asset paying a dividend yield of q

Parameters p, u, and d are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world

Mean: $e^{(r-q)\Delta t} = pu + (1-p)d$

Variance: $\sigma^2\Delta t = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t}$

A further condition often imposed is $u = 1/d$

Tree Parameters for asset paying a dividend yield of q (continued)

When Δt is small a solution to the equations is

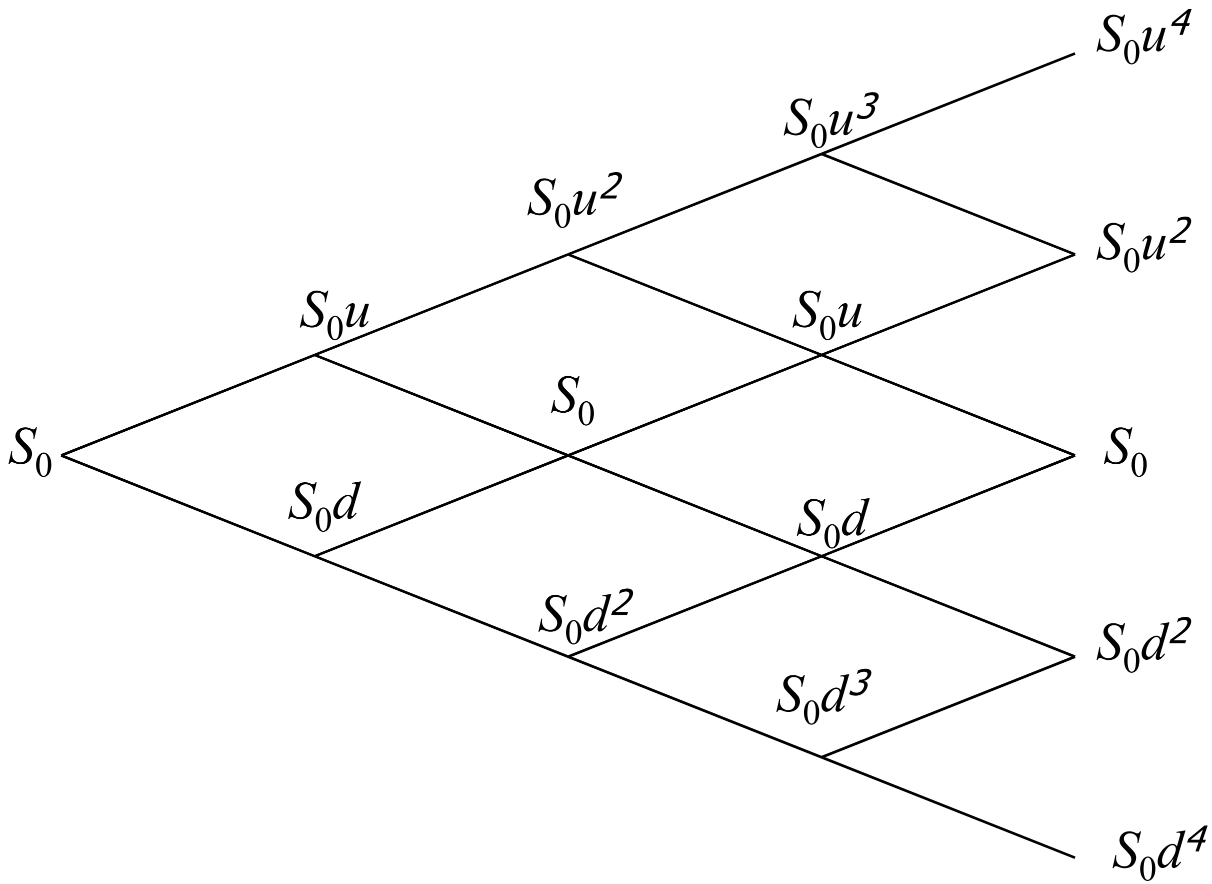
$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = e^{-\sigma \sqrt{\Delta t}}$$

$$p = \frac{a - d}{u - d}$$

$$a = e^{(r-q)\Delta t}$$

The Complete Tree



Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

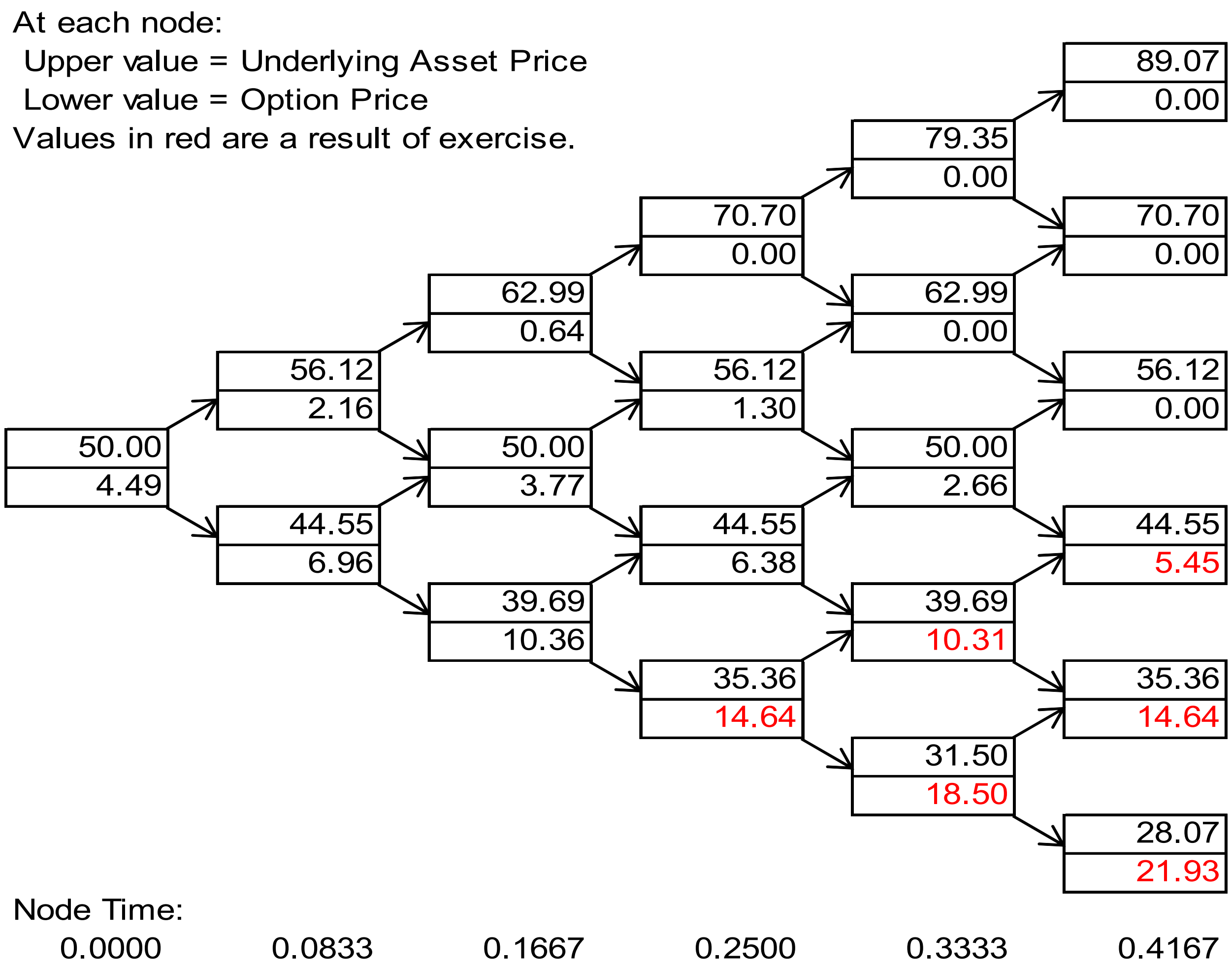
Example: Put Option

$$S_0 = 50; \; K = 50; \; r = 10\%; \; \sigma = 40\%;$$
$$T = 5 \text{ months} = 0.4167; \; \Delta t = 1 \text{ month} = 0.0833$$

In this case

$$a = e^{0.1 \times 1/12} = 1.0084$$
$$u = e^{0.4\sqrt{1/12}} = 1.1224$$
$$d = \frac{1}{u} = 0.8909$$
$$p = \frac{1.0084 - 0.8909}{1.1224 - 0.8909} = 0.5073$$

Example


At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Values in red are a result of exercise.
50.00
4.49
56.12
2.16
44.55
6.96
62.99
0.64
50.00
3.77
39.69
10.36
70.70
0.00
56.12
1.30
44.55
6.38
35.36
14.64
79.35
0.00
62.99
0.00
50.00
2.66
39.69
10.31
31.50
18.50
89.07
0.00
70.70
0.00
56.12
0.00
44.55
5.45
35.36
14.64
28.07
21.93
Node Time:
0.0000
0.0833
0.1667
0.2500
0.3333
0.4167

Calculation of Delta

Delta is calculated from the nodes at time Δt

$$\text{Delta} = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$

Calculation of Gamma

Gamma is calculated from the nodes at time $2\Delta t$

$$\Delta_1 = \frac{0.64 - 3.77}{62.99 - 50} = -0.24;\ \ \Delta_2 = \frac{3.77 - 10.36}{50 - 39.69} = -0.64$$

$$\text{Gamma} = \frac{\Delta_1 - \Delta_2}{11.65} = 0.03$$

=0.5(62.99-50)+0.5(50-39.69)

Calculation of Theta

Theta is calculated from the central nodes at times 0 and $2\Delta t$

$$\text{Theta} = \frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

or -0.012 per calendar day

Calculation of Vega

- We can proceed as follows
- Construct a new tree with a volatility of 41% instead of 40%.
- Value of option is 4.62
- Vega is

$$4.62 - 4.49 = 0.13$$

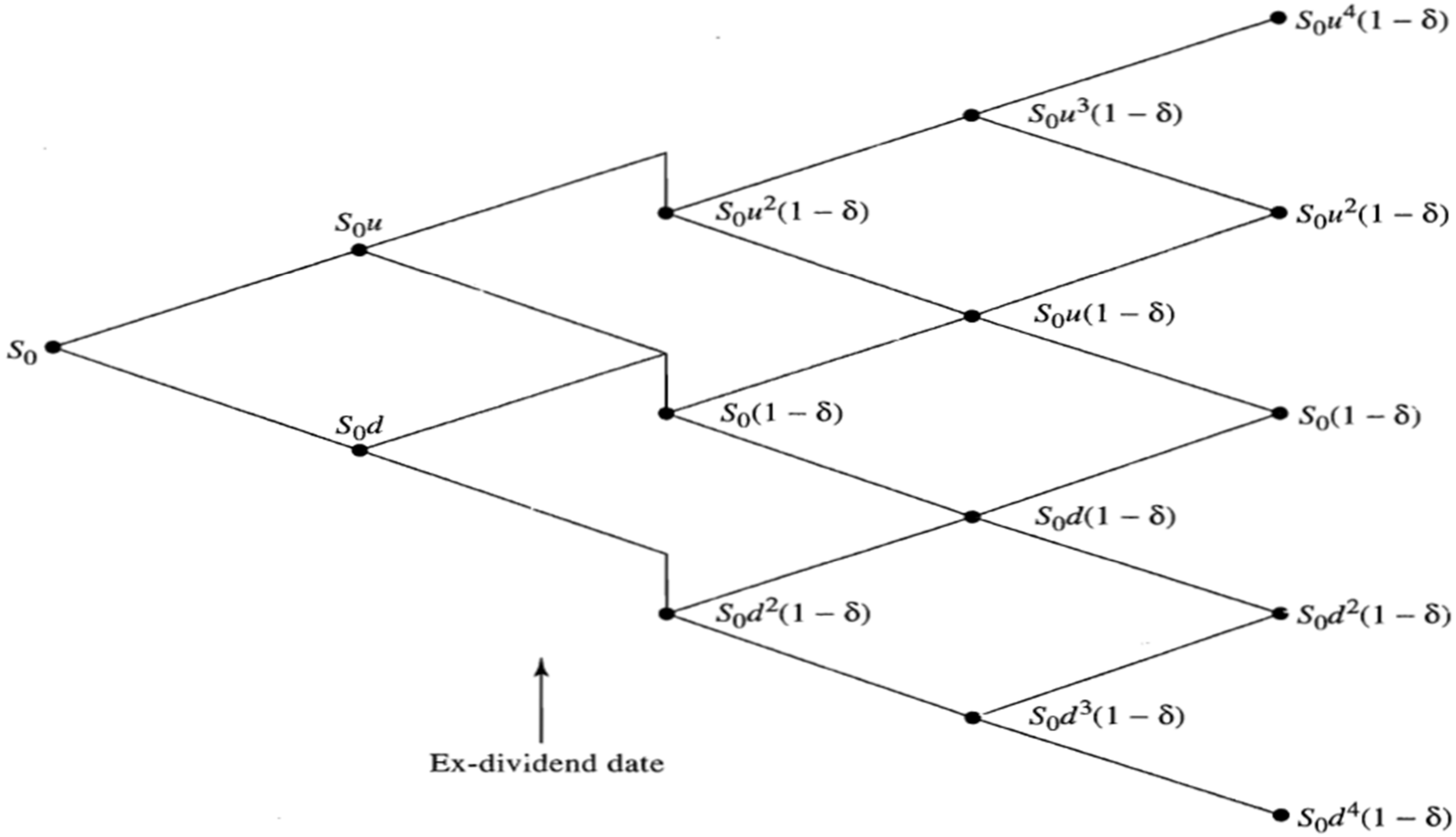
per 1% change in volatility

Trees for Options on Indices, Currencies and Futures Contracts

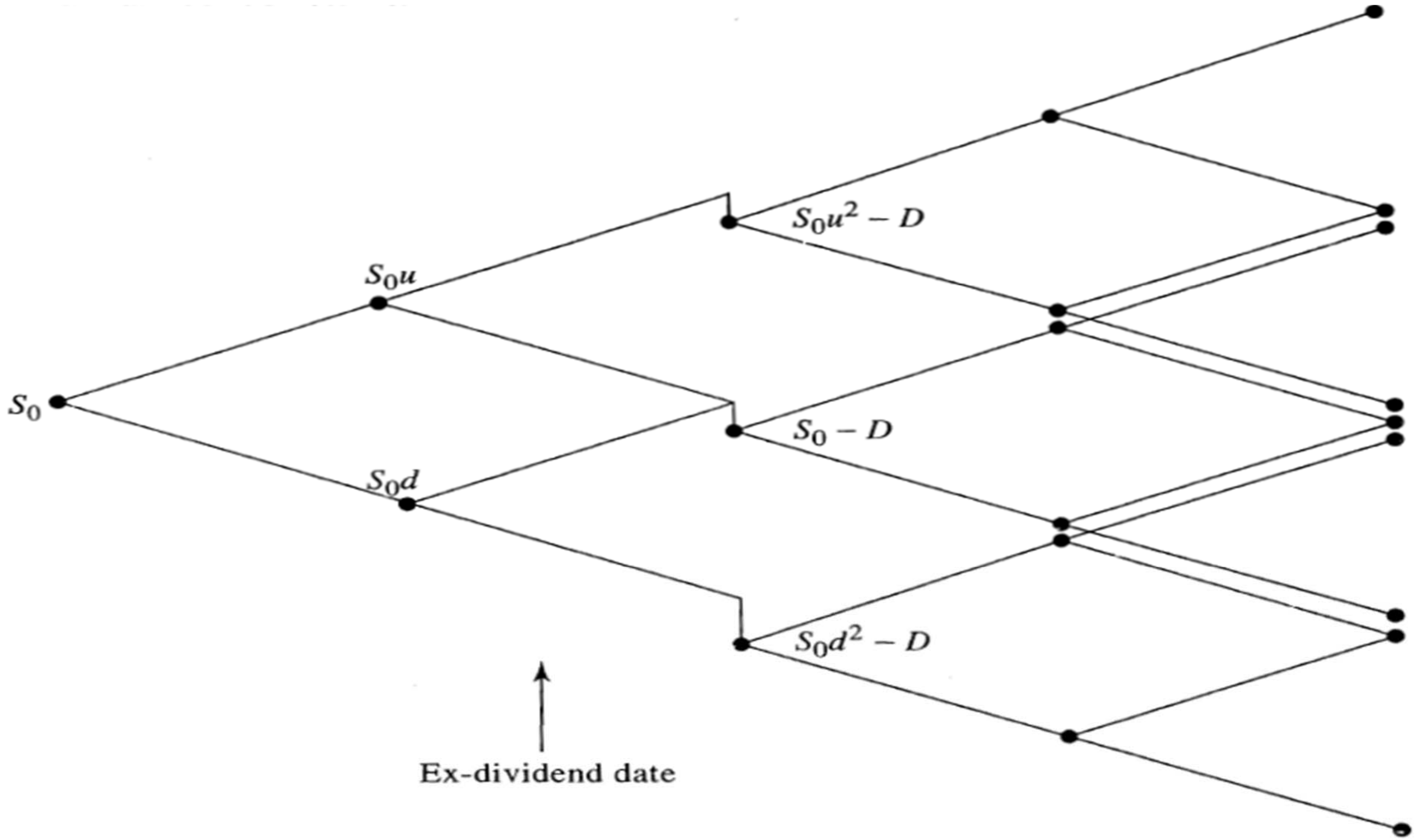
As with Black-Scholes-Merton:

- For options on stock indices, q equals the dividend yield on the index
- For options on a foreign currency, q equals the foreign risk-free rate
- For options on futures contracts $q = r$

Binomial Tree for Stock Paying Known Dividends Yield



Binomial Tree for Stock Paying Known Dividends



Binomial Tree for Stock Paying Known Dividends

- Procedure:
  - Construct a tree for the stock price less the present value of the dividends
  - Create a new tree by adding the present value of the dividends at each node
- This ensures that the tree recombines and makes assumptions similar to those when the Black-Scholes-Merton model is used for European options

Control Variate Technique

- Value American option, f_A
- Value European option using same tree, f_E
- Value European option using Black-Scholes – Merton, f_{BS}
- Option price $= f_A + (f_{BS} - f_E)$

Alternative Binomial Tree

Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

$$u = e^{(r-q-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}}$$

$$d = e^{(r-q-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$

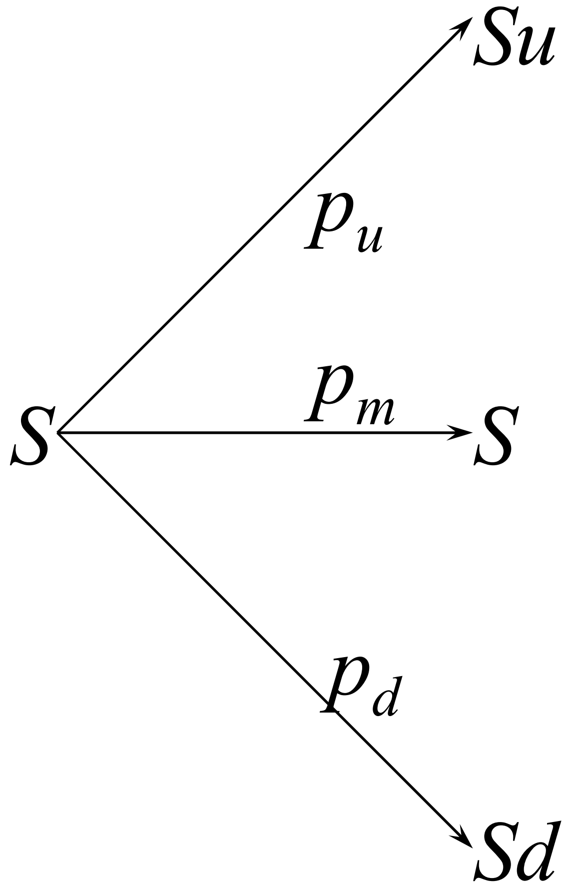
Trinomial Tree

$$u = e^{\sigma\sqrt{3\Delta t}} \qquad d = 1/u$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$p_m = \frac{2}{3}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$



Time Dependent Parameters in a Binomial Tree

- Making r or q a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time.

To make r and q (or r_f) a function of time in a Cox–Ross–Rubinstein binomial tree, we set

$$a = e^{[f(t)-g(t)]\Delta t} \tag{20.11}$$

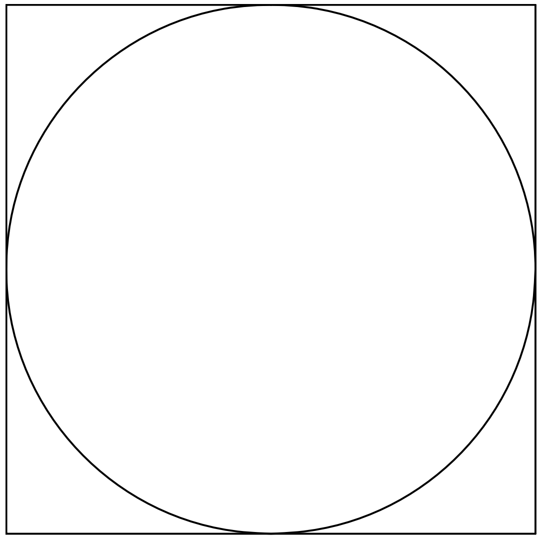
for nodes at time t, where $f(t)$ is the forward interest rate between times t and $t+\Delta t$ and $g(t)$ is the forward value of q (or r_f) between these times. This does not change the geometry of the tree because u and d do not depend on a. The probabilities on the

Time Dependent Parameters in a Binomial Tree

- We can make σ a function of time by making the lengths of the time steps inversely proportional to the variance rate. The values of u and d are then always the same and the tree recombines.

Monte Carlo Simulation and π

- How could you calculate π by randomly sampling points in the square?



Monte Carlo Simulation and Options

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoffs
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Sampling Stock Price Movements

- In a risk neutral world the process for a stock price is

$$dS = \hat{\mu} S\, dt + \sigma S\, dz$$

- We can simulate a path by choosing time steps of length Δt and using the discrete version of this

$$\Delta S = \hat{\mu} S\, \Delta t + \sigma S\, \varepsilon \sqrt{\Delta t}$$

where ε is a random sample from $\phi(0,1)$

A More Accurate Approach

Use

$$d\ \ln S = \left(\hat{\mu} - \sigma^2/2\right)dt + \sigma\, dz$$

The discrete version of this is

$$\ln S(t+\Delta t) - \ln S(t) = \left(\hat{\mu} - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

or

$$S(t+\Delta t) = S(t)\, e^{\left(\hat{\mu} - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

Extensions

When a derivative depends on several underlying variables we can simulate paths for each of them in a risk-neutral world to calculate the values for the derivative

Sampling from Normal Distribution

- In Excel =NORMSINV(RAND()) gives a random sample from $\phi(0,1)$

To Obtain 2 Correlated Normal Samples

- Obtain independent normal samples x_1 and x_2 and set

$$\varepsilon_1 = x_1$$
$$\varepsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

- Use a procedure known as Cholesky's decomposition when samples are required from more than two normal variables (see page 473)

Standard Errors in Monte Carlo Simulation

The standard error of the estimate of the option price is the standard deviation of the discounted payoffs given by the simulation trials divided by the square root of the number of observations.

Application of Monte Carlo Simulation

- Monte Carlo simulation can deal with path dependent options, options dependent on several underlying state variables, and options with complex payoffs
- It cannot easily deal with American-style options

Determining Greek Letters

For Δ:

1. Make a small change to asset price
2. Carry out the simulation again using the same random number streams
3. Estimate Δ as the change in the option price divided by the change in the asset price

Proceed in a similar manner for other Greek letters

Variance Reduction Techniques

- Antithetic variable technique（对偶变量技术）
- Control variate technique
- Importance sampling（虚值部分不用抽）
- Stratified sampling（分层抽样，抽样次数已知）
- Moment matching
- Using quasi-random sequences（平衡抽样）

Antithetic variable technique

- 在一次模拟运算中，计算两个期权价值：用通常方法计算得到f_1，改变计算 中所有的ε的符号得到f_2，这条模拟路径得到的期权价值是f_1和f_2的平均值

- 当一次模拟中的一个 值高于真实值时，另一个值必然偏低，反之亦然。从而大大降低方差。

Control variate technique

- 和二叉树模型中的控制方差技术相同，只是在蒙特卡罗中，对于两个类似的期权，必须使用相同的随机数流和相同的Δt平行地进行两次模拟

$$f_A = f_A^* - f_B^* + f_B$$

Importance sampling

- 适合于那些大部分路径对定价意义不大的情形，比如一个深度虚值的看涨期权，大部分路径上的终值为零。这时我们只选取那些标的资产的到期日价值大于其行权价格的路径，即重要路径为期权定价。这样等于缩小了样本空间，从而加速了收敛。

- 最后从重要路径中获得的贴现平均值还要再乘上（重要路径出现的概率），才能获得期权价值的最终估计值。

Stratified sampling

- 将市场变量在未来时刻的基本概率分布分为多个区间，并根据它的概率从每个间隔中抽样
- 假设存在10个可能性基本相等的区间，抽样方案就可以设计为每个间隔中抽取的样本各占10％
- 如果样本间隔的数量很多，我们就可以取每个区间内的均值或是中位数作为该区间的代表样本值。这样也可以提高模拟运算的效率
- 参见M. Curran, "Strata Gems," *RISK*, (March 1994), 70-71; B. Moro, "The Full Monte," *RISK*, (February 1985), 57-58.

Moment matching

- 对从标准正态分布中抽取的样本进行调整，使其一阶矩、二阶矩甚至高阶矩都匹配。

- 样本矩匹配法可以节省计算时间，但是由于所有的抽样值都要保存到模拟运算结束，存储负担很重

Using quasi-random sequences

- 在准随机序列抽样法中，每次抽样都试图填补之前已存在的样本之间的空缺，使得抽取的样本值总是能大致均匀地分布在整个概率空间中，使模拟的收敛速度得到改进。

Sampling Through the Tree

- Instead of sampling from the stochastic process we can sample paths randomly through a binomial or trinomial tree to value a derivative
- At each node that is reached we sample a randon number between 0 and 1. If it is between 0 and p, we take the up branch; if it is between p and 1, we take the down branch
- 参见D. Mintz, "Less is more," RISK, 10, 7 (July 1997), 42-45

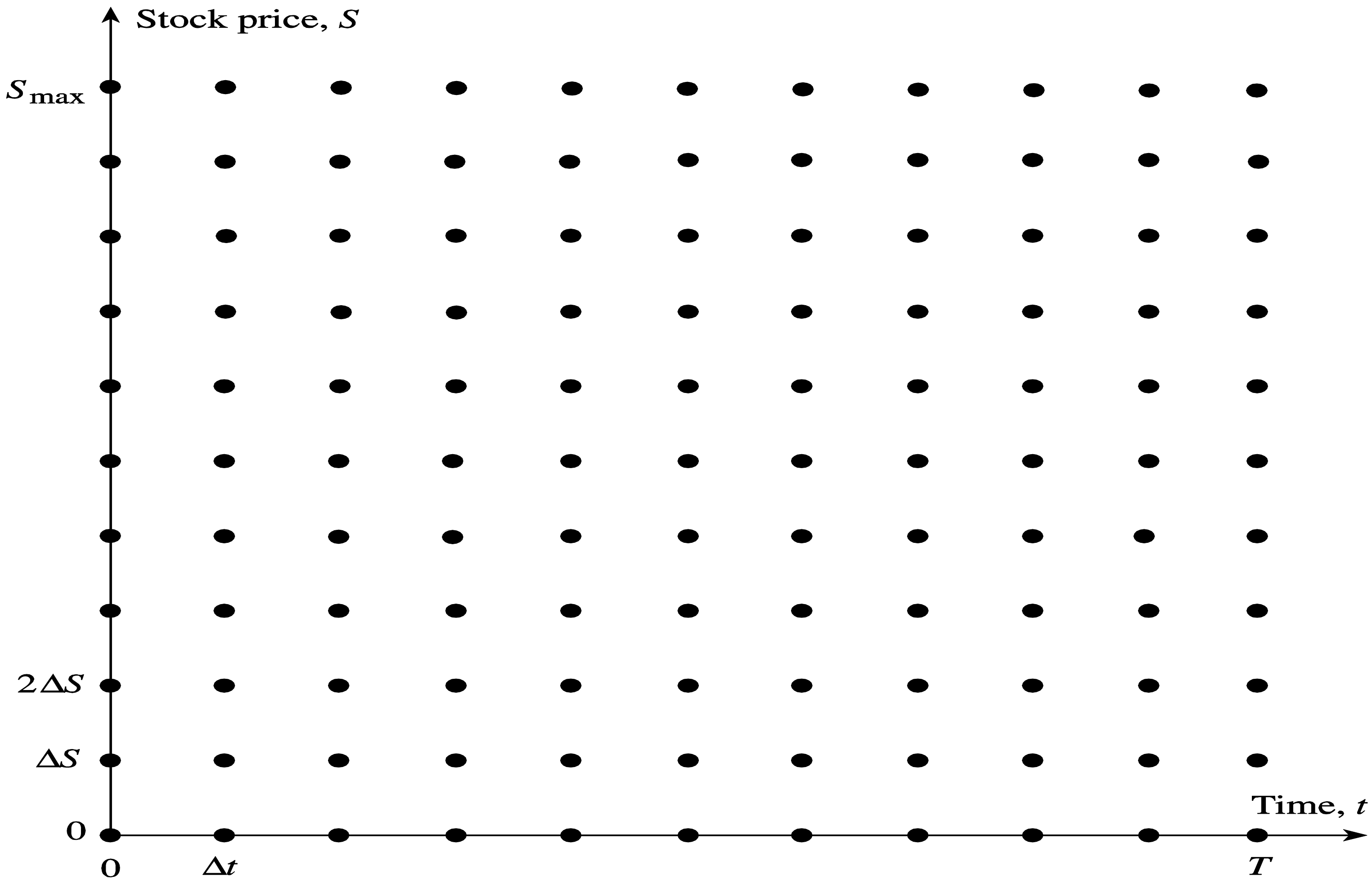
主要优缺点

- 主要优点：
  - 应用简单，无需深刻理解定价模型
  - 可给出标准误和置信区间
  - 适用情形广泛
    - 欧式衍生产品
    - 回报路径依赖
    - 回报取决于多个标的资产
- 主要缺点：
  - 难以处理提前执行的情形
  - 为了达到一定的精确度，一般需要大量的模拟运算

Finite Difference Methods

- Finite difference methods aim to represent the differential equation in the form of a difference equation
- We form a grid by considering equally spaced time values and stock price values
- Define $f_{i,j}$ as the value of f at time $i\Delta t$ when the stock price is $j\Delta S$

The Grid



Finite Difference Methods
(continued)

$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Set
$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \right) \Big/ \Delta S \qquad \text{or}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}$$

Implicit Finite Difference Method

Set $$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

to obtain for each node an equation of the form of the form:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}$$

显性有限差分法一方法1

- 假设 (i, j) 点的 $\frac{\partial f}{\partial S}$ 和 $\frac{\partial^2 f}{\partial S^2}$ 与 (i + 1, j) 的对应值相等，即

$$\frac{\partial f}{\partial S}=\frac{f_{i+1,j+1}-f_{i+1,j-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2}=\frac{f_{i+1,j+1}+f_{i+1,j-1}-2f_{i+1,j}}{\Delta S^2}$$

显性有限差分法 —方法1

- 相应的差分方程修改为

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

- 其中

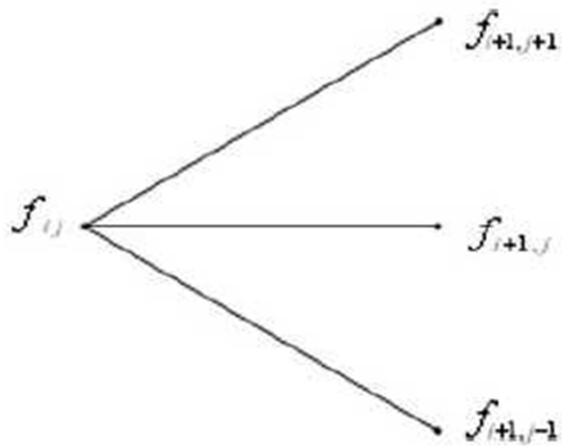
$$a_j^* = \frac{1}{1+r\Delta t}\left(-\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right)$$

$$b_j^* = \frac{1}{1+r\Delta t}\left(1-\sigma^2 j^2 \Delta t\right)$$

$$c_j^* = \frac{1}{1+r\Delta t}\left(\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right)$$

理解显性有限差分法

- 可以看出$a_j^* + b_j^* + c_j^* = \dfrac{1}{1 + r\Delta t}$，这说明某一时刻某格点的期权价值等于下一时刻相邻三个格点的期权价值风险中性期望值的现值。
- 显性有限差分法可以理解为从格点图外部推知内部格点期权价值的方法。



显性有限差分法—方法2

- 如果我们采用$\partial f/\partial t$的另一种定义，而保留$\partial f/\partial S$和$\partial^2 f/\partial S^2$的定义就可以得到另一种显性有限差分法。

令

$$\frac{\partial f}{\partial t}=\frac{f_{i,j}-f_{i-1,j}}{\Delta t}$$，将其代入 B-S-M 偏微分方程，整理可得

$$f_{i-1,j}=a^* f_{i,j-1}+b^* f_{i,j}+c^* f_{i,j+1}$$

$$a^*=-\frac{1}{2}(r-q)j\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t$$

其中，$b^*=1-r\Delta t-\sigma^2 j^2\Delta t$

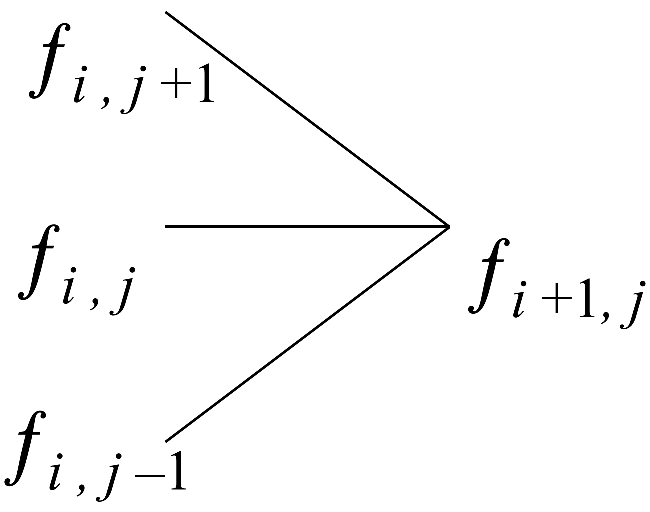
$$c^*=\frac{1}{2}(r-q)j\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t$$

从公式可以看出，$a^*+b^*+c^*=1-r\Delta t$。

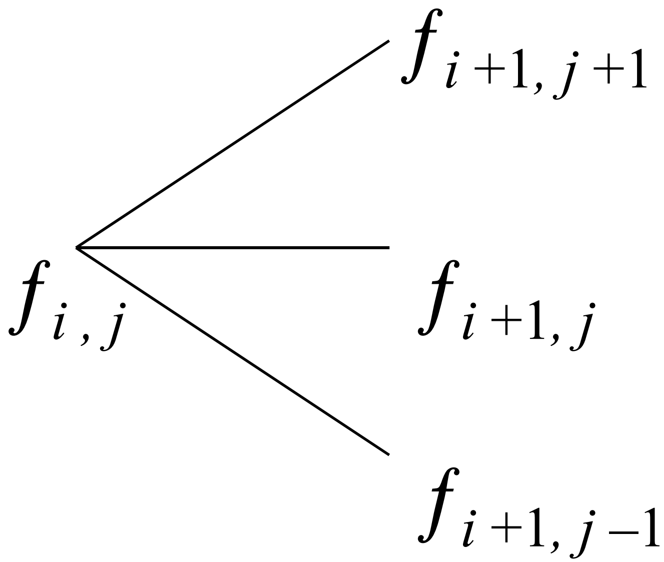
Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach
- The implicit finite difference method is equivalent to a multinomial tree approach

Implicit vs Explicit Finite Difference Methods



Implicit Method



Explicit Method

隐性和显性有限差分方法的比较

- 显性方法计算比较直接方便，无需像隐性方法那样需要求解大量的联立方程，工作量小，易于应用

- 但显性方法的三个“概率”可能小于零，导致了这种方法的不稳定，它的解有可能不收敛于偏微分方程的解。而隐性方法则不存在这个问题，它始终是有效的

有限差分方法和树图方法的比较

- 相同之处：都用离散的模型模拟资产价格的连续运动
- 不同之处
  - 树图方法中包含了资产价格的扩散和波动率情形；有限差分方法中的格点则是固定均匀的，只是参数进行了相应的变化，以反映改变了的扩散情形。
  - 有限差分方法比树图方法灵活
- 显性有限差分方法与三叉树图相当类似，但显性差分方法中的隐含概率可能小于零，这也是这一方法的主要缺陷。

有限差分方法的改进

- 变量置换：在使用有限差分方法时，人们常常把标的变量 S 置换为$Z=\ln S$。这样偏微分方程改为

$$\frac{\partial f}{\partial t}+\left(r-\frac{\sigma^2}{2}\right)\frac{\partial f}{\partial Z}+\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial Z^2}=rf$$

- Crank-Nicolson方法：显性有限差分方法和隐性有限差分方法的平均，需要用到六个点

$$f_{i,j}+f_{i-1,j}=a_jf_{i-1,j-1}+b_jf_{i-1,j}+c_jf_{i-1,j+1}+a_j^*f_{i,j-1}+b_j^*f_{i,j}+c_j^*f_{i,j+1}$$

- 需要解联立方程组从外向内推出内部格点的期权价值。在数学上可以借助于著名的托马斯算法求解这个方程组

- 尽管计算也相当复杂，但是Crank-Nicolson方法最大的优点在于它比一般的有限差分法有更高的精确度和无条件的稳定性，其成为在实践中广泛应用的一种有限差分方法

有限差分方法的应用

- 有限差分方法和树图方法相当近似的，可以解决相同类型的衍生证券定价问题，尤其是那些具有提前执行特征的期权

- 有限差分方法可以进一步推广到多个标的变量的情形，但超过三个变量时蒙特卡罗模拟方法较为有效

- 有限差分方法不善于处理期权价值取决于标的变量历史路径的情况

Any Questions？

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