Chp.4 Stochastic Discount Factor

Main Contents

- The Relationship between Law of One Price and Existence of Discount Factor;
- The Relationship Between No Arbitrage and Existence of Positive Discount Factor;
- An Alternative Formula to Compute the Discount Factor in Discrete and Continuous Time.

4.1 law of one price and Existence of a Discount factor

Assumptions

- **A1:(Portfolio formation):** $x_1, x_2 \in \underline{X} \Rightarrow ax_1 + bx_2 \in \underline{X}$
for any real a and b.
- Remark: It's an important and restrictive simplifying assumption. short sales constraints, leverage limitations, and so on.
- **A2:(Law of one price, Linearity):**

$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2)$$

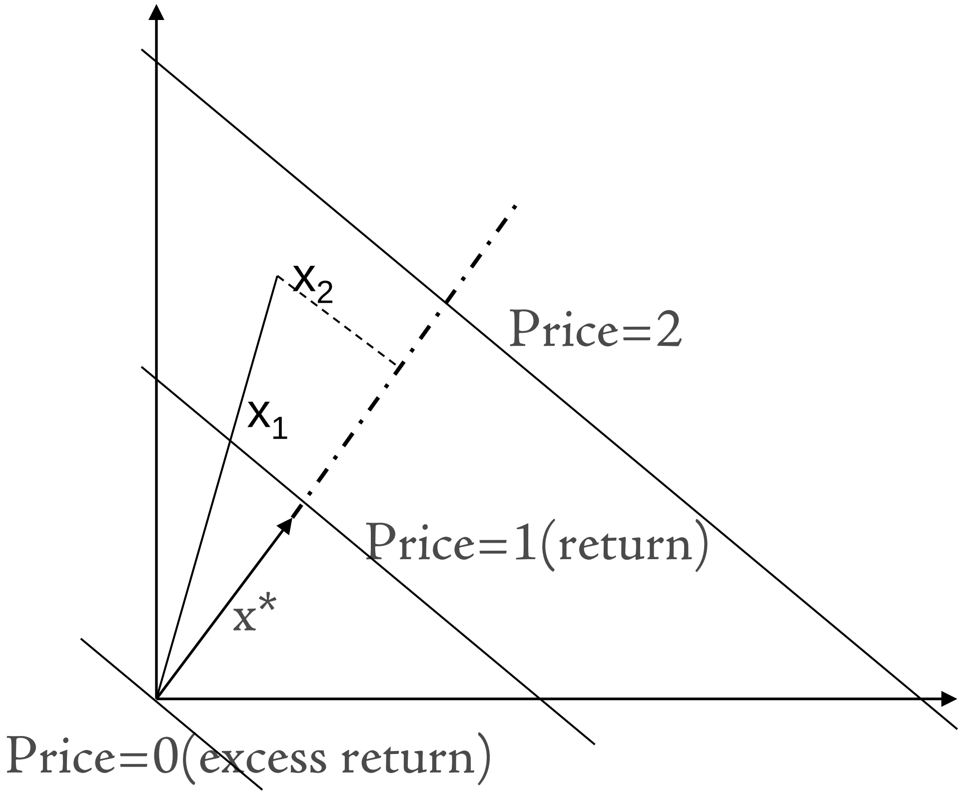
- Remark: if the payoff of asset A is the same as that of asset B in any case, then price of A=price of B. happy meal theorem. It rules out bid/ask spreads.不考虑流动性。

Theorem 1

- Given free portfolio formation A1, and the law of one price A2, there exists a unique $x^* \in \underline{X}$ such that p(x)=E(x*x) for all $x \in \underline{X}$.

Geometric Proof 1

- 一价定律＝线性价格函数。
- 线性价格函数⟹等价线如下图所示。假设回报空间是二维的。
- 根据p＝0等价线可知x*与之正交。（存在）（注意我们定义 $E(XY)=X\circ Y$，因此求内积时要乘以概率 ）



Geometric Proof 2

- 用x*为p=1等价线上的任一证券x_1定价可确定x*的长度。即：

$$\left|proj(x_1|x^*)\right| \times \left|x^*\right| = 1 \Rightarrow \left|x^*\right| = 1/\left|proj(x_1|x^*)\right|$$

- 给定任意证券x_2，它与0连线（或延长线）与p＝1等价线相交于x_1,即x_2 =a x_1。从图上可以看出，用x*定价可得p(x_2)=ap(x_1),符合一价定律。

Algebraic Proof

- Suppose the basis payoffs (after pruning redundant rows of x)

  $x=[x_1, x_2, ..., x_N]'$，注意：x在这里是矩阵

- Then we want to find a discount factor x* in payoff space，so it must be of the form

$$x^* = c'x$$

- 对于任意的证券组合$a'x$,我们用x*来定价得：

$$p(a'x) = E((c'x)(a'x)) = c'aE(x'x)$$

由一价定律可知$p(a'x)=a'p$

$$\therefore E(x'x)c = p \Rightarrow c = E(xx')^{-1}p$$

$$x^* = p'E(xx')^{-1}x$$

- 由于x*对于任意证券都一样，因此是唯一的。

Other discount factors

- The discount factor in payoff space X is unique.
- There are many other discount factors m not in X. (unless the market is complete).
- If $p=E(mx)$, then $p=E[(m+\varepsilon)x]$ for any ε orthogonal to x, $E(\varepsilon x)=0$.
- Any discount factor m can be represented as $m=x^*+\varepsilon$, with $E(\varepsilon x)=0$.
- The pricing implication of any discount factor m for a set of payoff X are the same as those of projection m on X.

$$p = E(mx) = E[(proj(m \mid X) + \varepsilon)x] = E[proj(m \mid X)x]$$

- $proj(m \mid X)$ is called the mimicking portfolio for m.

Theorem 2

- The existence of a discount factor implies the law of one price
- Proof: if $x+y=z$,and there is a discount factor, then $p(x+y)=E(m(x+y))=E(mz)=p(z)$

4.2 No Arbitrage and Positive Discount Factors

Definition: No arbitrage

- D1:Every payoff x that is always nonnegative (almost surely), and positive with some positive probability, has positive price.
- D2:If x>=y almost surely and x>y with positive probability, then p(x)>p(y).

Theorem3: m>0 imply No arbitrage

- Proof:
  - For x>=0 and in some states x>0.
  - Because m>0(positive in every state).
  - P=E(mx)>0

No arbitrage implies a m>0

- 直觉：
  - 无套利意味着正象限payoff的价格严格为正.p=0线将正负价格区域分割开来。为了使正负价格区域不交叉，等价线必须经过0和第2、4象限，因此m必须经过0指向第一象限。

Theorem4:No arbitrage implies a m>0，可以对回报空间的任何x定价

- 证明：由于无套利蕴含着一价定律，也就意味着存在随机折现因子，故仅需证明m为正的。
- 联合(-p(x),x) 形成s+1维空间 R^{s+1}中的向量。令M表示所有的数对(-p(x),x) 构成的集合。

$$M=\left\{(-p(x),x);x\in \underline{X}\right\}$$

- 由一价定律，M仍是一个线性空间。
- 无套利意味着M的元素（s+1维向量）不能够全部由正的分量组成。如果x是正的，那么- p(x),一定为负（无套利保证的）。这样，超平面M就与正的向量空间 $\mathbf{R}_{+}^{s+1}$ 只相交于原点。

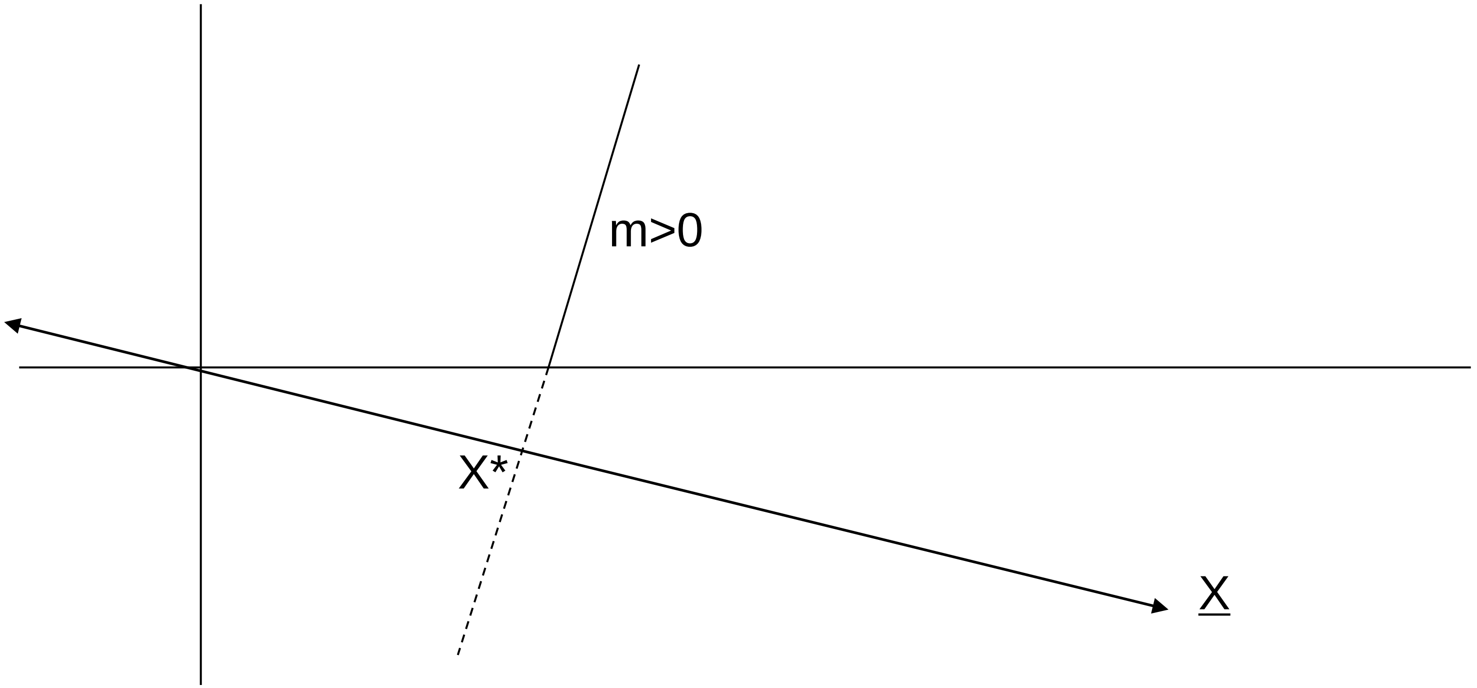
- 这样就存在一个函数F：$\mathbf{R}^{s+1} \Rightarrow \mathbf{R}$使得对于$(-p(x),x) \in M$的点 F(-p,x)=0， 并且 除原点外的$(-p(x),x)$的点F(-p,x)>0 （由超平面分离定理保证的）。
- 由于可以采用向量的内积来表示任何的线性函数，并且存在向量（1，m）使得

$$F(-p,x)=(1,m)\cdot(-p,x)=-p+m\cdot x$$
$$or\ \ -p+E(mx)$$

- 由于对所有$(-p(x),x)$>0的点F(-p,x)都是正的，所以m必须是正的。
- 在连续的情况下，可以由凸集分离定理和Riesz表示定理同样得到结论

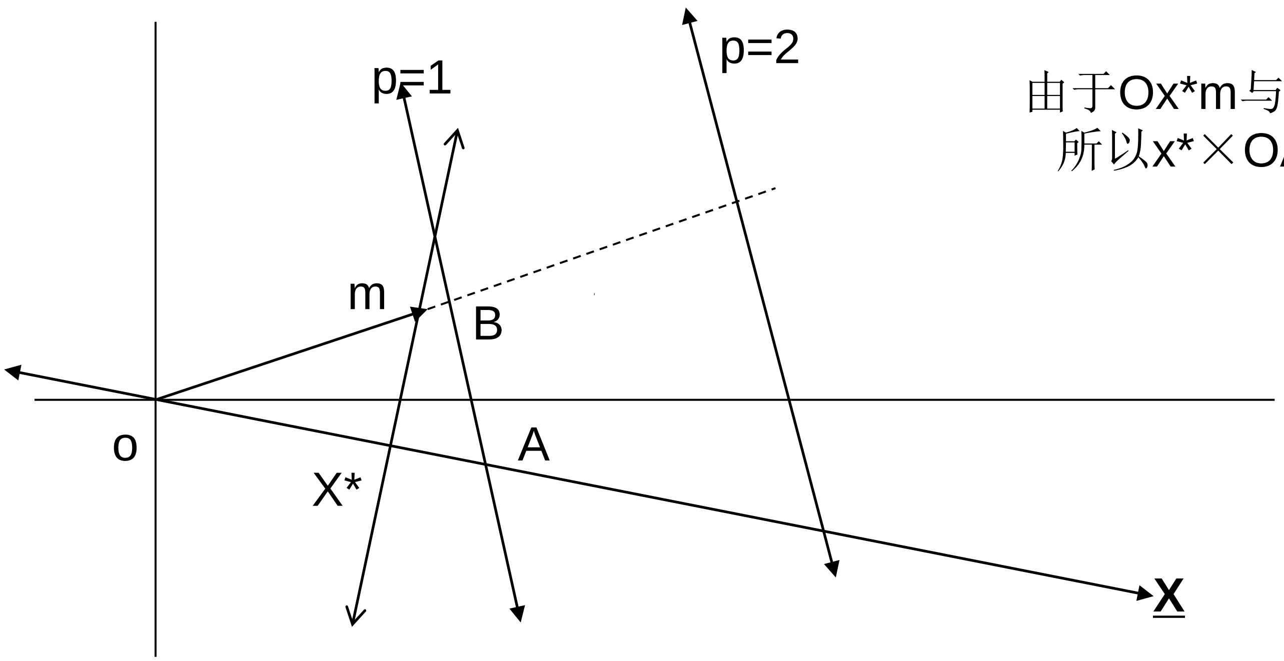
Other discount factors

- 从经济意义上讲，m应该为正。但m在回报空间中的投影不一定为正.
- In incomplete market, even x^* need not be positive.



Arbitrage-free extension of prices

- Each particular choice of m>0 induces an arbitrage-free extension of prices on X to all contingent claims



由于Ox*m与OBA相似，
所以x*×OA=OB×m

No arbitrage and the law of one price

- No arbitrage is more strict than the law of one price.
- No arbitrage implies the law of one price, but not vice versa.

Why no arbitrage is more strict than law of one price?

- Law of one price implies the same payoff has the same price, but does not consider the situation of different payoffs. For example, if payoff A>payoff B in any case, under the law of one price, p(A)<p(B) may hold. This implies arbitrage opportunity.
- No arbitrage implies positive payoff has positive price, which includes the law of one price.

4.3 an alternative formula, and x* in continuous time

Alternative fromula

- $$x^* = E(x^*) + [p - E(x^*)E(x)]' \sum{}^{-1} (x - E(x))$$
- Proof:

$$E(x^* x) = E\{[E(x^*) + (p - E(x^*)E(x))' \sum{}^{-1} (x - E(x))]x\}$$
$$= E(x^*)E(x) + (p - E(x^*)E(x))' \sum{}^{-1} E\{[x - E(x)]x\}$$
$$= E(x^*)E(x) + (p - E(x^*)E(x))' \sum{}^{-1} \sum = p$$

Alternative formula(2)

- If a risk-free rate is traded, and the payoff space consists solely of excess returns(p=0), then we have:

$$x^* = \frac{1}{R^f} - \frac{1}{R^f} E(R^e)' \Sigma^{-1} (R^e - E(R^e)); \Sigma \equiv \text{cov}(R^e)$$

X* in continuous time

- Similarly, we can get

$$\frac{d\Lambda^*}{\Lambda^*} = -r_f dt - (\mu + \frac{D}{p} - r_f)\sum{}^{-1}\sigma dz$$

- Proof:

$$假设:\frac{dp}{p} = \mu\ dt + \sigma\ dz, \frac{d\Lambda^*}{\Lambda^*} = -r^f dt - \sigma^\Lambda dz$$

$$E_t\frac{dp}{p} + \frac{D}{p}dt - r_f dt = (\mu + \frac{D}{p} - r_f)dt = -E_t(\frac{d\Lambda}{\Lambda}\frac{dp}{p}) = \sigma\sigma_\Lambda dt,$$

$$\sigma\sigma_\Lambda = (\mu + D/p - r_f),$$

$$\sum \sigma_\Lambda = (\mu + D/p - r_f)\sigma',$$

$$\sigma_\Lambda = \sum{}^{-1}(\mu + D/p - r_f)\sigma' = (\mu + D/p - r_f)'\sum{}^{-1}\sigma$$

Other discount factors in continuous time

- Λ^* plus orthogonal noise will also act as a discount factor:

$$\frac{d\Lambda}{\Lambda} = \frac{d\Lambda^*}{\Lambda^*} + dw; E(dw) = 0; E(dzdw) = 0.$$

重要结论（1）

- 在完全市场中，m只有一个，且严格为正。
- 在不完全市场中，即使处于无套利均衡状态，m在回报空间中的投影也可能为负。
- 在不完全市场中，新产品（只要不是原有产品的线性复制品）可以使市场趋于完全。但若没有其他信息，该产品就无法准确定价，但可以确定价格区间。

重要结论（2）

- 在完全市场中，等价鞅测度是唯一的。概率测度可以自由转换。而在不完全世界中，等价鞅测度不是唯一的。
- 在不完全世界，给定风险中性概率和任一资产（包括无风险资产）的价格，就可以求出在这种测度下任意新产品的价格。因为知道风险中性概率就知道所有状态价格之比，再利用已知价格就可算出所有状态价格，从而为所有资产定价。风险中性概率决定相对价格，该证券价格决定绝对价格。$\pi^*(s) = pc(s) / E(m)$
- 不完全世界中，即使我们知道现实的概率，和N（N小于S）种证券价格，我们仍无法对任意新产品定价。

The End